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LETTER TO THE EDITOR

The scattering of two Yang–Mills plane waves

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Abstract. We consider the head-on collision of two sandwich plane waves in non-abelian gauge theory. A classical solution of the Yang–Mills field equations is obtained describing the scattering of weak plane waves. We show that after the collision the gauge field acquires a new inductive aspect due to the nonlinear interaction between these waves.

Maxwell's electrodynamics predicts that light will simply pass through light without effect. On the other hand in general relativity space–time acquires mass and angular momentum aspects through the scattering of two gravitational waves. Furthermore the focusing of each gravitational plane wave by the gravitational field of the other one eventually brings about a space–time singularity (Penrose 1965, Szekeres 1970, 1971, Khan and Penrose 1971, Nutku and Halil 1977). It will be of interest to consider the scattering of two plane waves in classical Yang–Mills theory where in the abelian limit there will evidently be no interaction between the waves, but in general we can expect the non-abelian character of the field to reproduce some results of general relativity. Hence we shall now obtain the reduced field equations and formulate the boundary conditions appropriate to the problem of colliding Yang–Mills plane waves.

Plane wave solutions of non-abelian gauge fields were first obtained by Coleman (1977). He showed that the one-form

$$\boldsymbol{A} = [\boldsymbol{f}(u)x + \boldsymbol{g}(u)y + \boldsymbol{h}(u)]\,\mathrm{d}u \tag{1}$$

is a solution of the Yang–Mills field equations where $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are Lie algebra valued arbitrary functions of the retarded time coordinate u. For definiteness we shall choose the gauge group to be SU(2) and boldface letters will denote isovectors. Trautman (1980) has pointed out that if $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ belong to an abelian Lie algebra, then translations in the xy plane are symmetries.

We shall start by considering a specialisation of equation (1) to a sandwich plane wave. Let us refer to figure 1 which depicts Minkowski space divided into four regions. In regions I, II and III there will be sandwich plane waves travelling in positive and negative z directions respectively, while their interaction will take place in region IV. The potential one-form will vanish in region I

$$\boldsymbol{A}^{\mathrm{I}} = 0 \tag{2}$$

and henceforth Roman numeral superscripts will refer to the various regions. The sandwich wave travelling in (say) the $+z$ direction will start with a shock at $u = u_1$ to be followed by an equal and opposite shock at $u = u_2$ where u_1 and u_2 are arbitrary

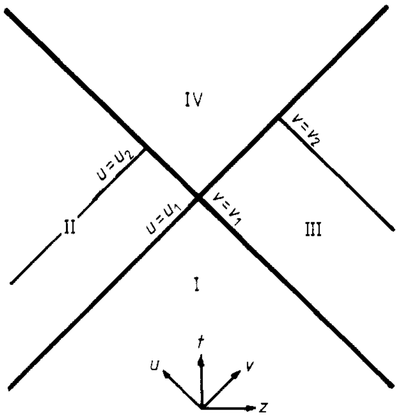

Figure 1. Regions of Minkowski space which are used in formulating the colliding waves problem. Each point in the diagram represents a plane.

constants satisfying $u_1 < u_2$. This situation is described by the potential one-form

$$\boldsymbol{A}^{\mathrm{II}} = \varepsilon \boldsymbol{n}_0 \mathscr{A}(u)\,\mathrm{d}x, \tag{3}$$

$$\mathscr{A}(u) = (u_2 - u_1)^{-1}[(u-u_1)\theta(u-u_1) - (u-u_2)\theta(u-u_2)], \tag{4}$$

where θ is the Heaviside unit step function, $\boldsymbol{n}_0$ is a constant isovector and ε is a constant parameter which can be regarded as the amplitude of the wave. The expression for $\mathscr{A}(u)$ could have been smoothed out by choosing suitable functions, but the use of distributions simplifies the problem and henceforth we shall assume the Yang–Mills field equations to hold in the sense of distributions. Equations (2)–(4) describe a sandwich wave since the field constructed from these expressions vanishes for $u < u_1$ and $u > u_2$. In particular, the limit $u_2 \to u_1$ yields an *impulsive* wave (Penrose 1972), that is, the field two-form F becomes

$$\lim_{u_2 \to u_1} \boldsymbol{F}^{\mathrm{II}} = \varepsilon \boldsymbol{n}_0 \delta(u-u_1)\,\mathrm{d}u \wedge \mathrm{d}x, \tag{5}$$

where δ denotes the Dirac delta function. These sandwich plane waves form a subclass of the non-abelian waves. They are plane abelian waves pointing in a fixed direction in isospin space. Equation (3) is obtained from equation (1) for the choice $\boldsymbol{g} = \boldsymbol{h} = 0$ together with a gauge transformation which enables us to eliminate the undesirable dependence of the potential on the ignorable coordinates x and y defined along planes of symmetry. A similar specialisation of equation (1) will describe a sandwich plane wave travelling in the negative z direction

$$\boldsymbol{A}^{\mathrm{III}} = \eta \boldsymbol{m}_0 \mathscr{A}(v)\,\mathrm{d}y \tag{6}$$

where v is the advanced time coordinate, $\boldsymbol{m}_0$ is a constant isovector and η is another parameter standing for the amplitude of the wave.

The scattering of these two sandwich plane waves will be formulated as a characteristic initial value problem for the Yang–Mills field equations with Cauchy data specified on a pair of intersecting null surfaces. Thus we shall ask for a solution of the field equations in region IV such that on $u = u_1$ and $v = v_1$ the potentials are given by (3) and (6). The junction conditions across the various regions will be the same as those in Maxwell's theory. A suitable ansatz for the potential one-form in the interaction region is

$$A^{\mathrm{IV}} = \boldsymbol{n}(u, v)\,\mathrm{d}x + \boldsymbol{m}(u, v)\,\mathrm{d}y \tag{7}$$

which is suggested by (3) and (6). We note here that the transformation of the potential into the form of (3) in region II was an essential simplification without which both waves could not have been represented by this ansatz. The field two-form constructed from (7) is given by

$$F^{\mathrm{IV}} = \boldsymbol{n}_u\,\mathrm{d}u \wedge \mathrm{d}x + \boldsymbol{n}_v\,\mathrm{d}v \wedge \mathrm{d}x + \boldsymbol{m}_u\,\mathrm{d}u \wedge \mathrm{d}y + \boldsymbol{m}_v\,\mathrm{d}v \wedge \mathrm{d}y + \boldsymbol{n} \times \boldsymbol{m}\,\mathrm{d}x \wedge \mathrm{d}y \tag{8}$$

where here and in the following subscripts denote partial derivatives. Then, the Yang–Mills field equations

$$\mathrm{d}^*\boldsymbol{F} + \boldsymbol{A} \wedge {}^*\boldsymbol{F} = 0, \tag{9}$$

where $\wedge$ denotes both wedge product of forms and cross product in isospin space, reduce to

$$\boldsymbol{n} \times \boldsymbol{n}_u + \boldsymbol{m} \times \boldsymbol{m}_u = 0, \qquad \boldsymbol{n} \times \boldsymbol{n}_v + \boldsymbol{m} \times \boldsymbol{m}_v = 0, \tag{10a, b}$$

$$2\boldsymbol{n}_{uv} = \boldsymbol{m} \times (\boldsymbol{n} \times \boldsymbol{m}) = \boldsymbol{n}|m|^2 - \boldsymbol{m}(\boldsymbol{m} \cdot \boldsymbol{n}), \tag{10c}$$

$$2\boldsymbol{m}_{uv} = \boldsymbol{n} \times (\boldsymbol{m} \times \boldsymbol{n}) = \boldsymbol{m}|n|^2 - \boldsymbol{n}(\boldsymbol{n} \cdot \boldsymbol{m}). \tag{10d}$$

In order to generalise these equations to another gauge group the cross product must be understood as $(n \times m)^i = C^i_{jk} n^j m^k$ where C^i_{jk} are the structure constants.

Equations (10) possess an invariance property which can be used to simplify them without loss of generality. Namely, a rotation on the xy plane can be compensated by a rotation of $\boldsymbol{n}$ and $\boldsymbol{m}$ in isospin space so that (7) remains invariant. We can verify that equations (10) are indeed invariant under the transformation

$$\boldsymbol{n} \to \boldsymbol{n} \cos\alpha - \boldsymbol{m} \sin\alpha, \qquad \boldsymbol{m} \to \boldsymbol{n} \sin\alpha + \boldsymbol{m} \cos\alpha, \tag{11}$$

and therefore given any two vectors $\boldsymbol{n}, \boldsymbol{m}$ we can always make them orthogonal to each other

$$\boldsymbol{n} \cdot \boldsymbol{m} = 0 \tag{12}$$

provided that they were not parallel to start with. These vectors will be parallel only for an abelian gauge field which is equivalent to a Maxwell field. In Maxwell's theory proper, only two scalar functions n, m will survive and they will satisfy

$$n_{uv} = 0, \qquad m_{uv} = 0, \tag{13}$$

in place of equations (10). These equations have the familiar solution that in the interaction region waves travelling in the positive and negative z directions will add without interference from each other. Thus for the interesting case of non-abelian gauge field we can always require that n, m in (7) satisfy (12) which simplifies (10c) and (10d). Further, (10a, b) can be satisfied by the choices

$$\boldsymbol{n} = \boldsymbol{n}_0 n, \qquad \boldsymbol{m} = \boldsymbol{m}_0 m, \tag{14}$$

where $\boldsymbol{n}_0, \boldsymbol{m}_0$ are constant orthogonal vectors and n, m are scalar functions of u, v. These specialisations are also consistent with the boundary conditions since the limiting forms of the potential in regions II and III can be obtained from (7) and (14). The Yang–Mills field equations now reduce to the pair

$$2n_{uv} = m^2 n, \qquad 2m_{uv} = n^2 m, \tag{15}$$

and it is not possible to obtain a simpler expression of the field equations without introducing *ad hoc* restrictions. Finally, (15) can be derived from a variational principle with the Lagrangian density

$$\mathscr{L} = 2n_u n_v + 2m_u m_v + n^2 m^2 \tag{16}$$

which is an interesting two-dimensional model field theory.

There exists a solution of this system of equations which satisfies the required boundary conditions. We have not been able to find the exact solution and had to be content with an approximate solution where

$$\varepsilon \ll 1, \qquad \eta \ll 1, \tag{17}$$

and the colliding waves are initially weak waves. This solution can be written in the form

$$n = \varepsilon\mathscr{A}(u) - \tfrac{1}{12}\varepsilon\eta^2\mathscr{B}(u)\mathscr{C}(v) + \ldots, \qquad m = \eta\mathscr{A}(v) - \tfrac{1}{12}\eta\varepsilon^2\mathscr{C}(u)\mathscr{B}(v) + \ldots, \tag{18}$$

where in addition to $\mathscr{A}(u)$ given by (4) we have defined

$$\mathscr{B}(u) = (u_2 - u_1)^{-1}[(u-u_1)^2\theta(u-u_1) - (u-u_2)^2\theta(u-u_2)],$$

$$\mathscr{C}(u) = (u_2-u_1)^{-2}\{[u^2 + (u_1 - 3u_2)u - (2u_1 - 3u_2)u_1](u-u_1)\theta(u-u_1) - (u-u_2)^3\theta(u-u_2)\}. \tag{19}$$

In the above expressions we have grouped the terms in such a way that the satisfaction of equations (15) throughout Minkowski space is manifest. Higher-order terms can be calculated in a straightforward manner.

In this solution we can recognise an important property of colliding plane waves in non-abelian gauge theory. The interaction between the plane waves results in only a slight modification of the wavefronts which pass through each other and continue on. But the nonlinearity of this interaction gives rise to a new phenomenon which is similar to the situation in general relativity. Namely, initially the gauge field consisted only of radiative parts but after the collision it acquires a new inductive aspect as well. Using (18) and (19) in (8) we find that the field two-form has radiative aspects corresponding to retarded and advanced waves, but in addition it also carries an inductive aspect which is given by the last term in (8). At the lowest order the inductive term grows monotonically with increasing u and v; however, we need the exact solution in order to find out whether or not it dies off long after the collision has taken place.

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